INFLUENCE OF CAPILLARY FORCES UPON THE COEFFICIENT OF CONTRACTION OF A JET

(VLIIANIE KAPILIARNYKH SIL NO KOEFFITSIENT SZHATIIA STRUI)

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The first treatment of the problems of jet flows with consideration of the forces of gravity and of surface tension is due to Zhukovskii [1]. It turned out that from the mathematical point of view both problems were similar. Zhukovskii produced exact solutions of one example for the case of a heavy fluid and another example (flow past a gas bubble between two walls) for the case with consideration of surface tension. Exact solutions of the problems of jet flows of a heavy fluid were also obtained subsequently to Zhukovskii's paper by others (for example, N. Bervi, Richardson). The only other paper dealing with the problem of jet flow with consideration of capillary forces known to me, besides the paper by Zhukovskii, is the paper by McLeod [2]. McLeod, evidently, did not know about Zhukovskii's paper [1] and using a new method investigated a particular case of the latter's problem.

Recently Voronets [3] has suggested the use of a method of perturbations for the solution of jet problems of a heavy fluid. The work of Voronets has been continued by Gurevich and Pykhteev [4]. In this paper an attempt is made to apply the method of small perturbations to the solution of jet problems with consideration of capillary forces in order to reveal more precisely the possibilities of the jet theory. The author also discusses the question of the superposition of linearized capillary waves upon jet flows. Inasmuch as the work is of an exploratory character, the author has limited himself to the investigation of the above problems, using as an example one of the simplest problems in jet theory.

1. Flow configuration. We shall investigate the plane problem of the efflux of a symmetrical jet of a weightless ideal incompressible fluid from an opening in a wall. In the derivation of the boundary conditions we shall take into account the forces of surface tension.

Consider one half of the jet-flow region and substitute a solid wall for the axis of symmetry which coincides with the x-axis CA (Fig. 1). The total width of the opening equals $2l$. Let the thickness of the entire jet be 2δ. The most important characteristic of the flow is the coefficient of contraction of the jet $k = \delta/l$.

2. Boundary conditions on the free surface. Let p_1 and p be the pressures in the atmosphere and in the fluid, respectively. We denote by R the radius of curvature of the free surface and by α the coefficient of surface tension. Then, we have the known relationship

$$p_1 = p + \frac{\alpha}{R} \tag{2.1}$$

If θ is the angle between the velocity and the x-axis, $w = \phi + i\phi$ the complex potential function, ds the differential arc length, v the velocity vector and v_0 the velocity vector at infinity at the point A, then

$$\frac{1}{R} = \frac{d\theta}{ds} = \frac{v\,d\theta}{d\varphi} \tag{2.2}$$

The curvature of the free surface of the jet is directed upward, hence $R > 0$. Bernoulli's integral yields

$$p - p_0 = \frac{\rho}{2}(v_0^2 - v^2) \tag{2.3}$$

where p_0 is the pressure at infinity at the point A. If we limit ourselves to the case when capillary waves are not present, then we have $p_0 = p_1$, i.e. at infinity $R = \infty$. From (2.1), (2.2) and (2.3) we have [1]

$$\alpha v \frac{d\theta}{d\varphi} = \frac{\rho}{2}(v^2 - v_0^2) \tag{2.4}$$

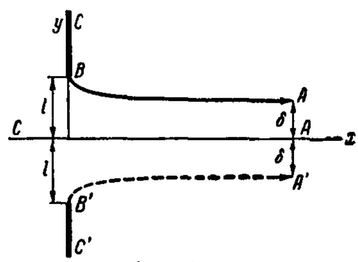

Fig. 1.

We define the flow in the jet above the x-axis by $q = v_0\delta$, and shall introduce nondimensional quantities V and a defined as follows:

$$V = \frac{v}{v_0}, \qquad a = \frac{\alpha}{\rho v_0 q} = \frac{\alpha}{\rho v_0^2 \delta} \tag{2.5}$$

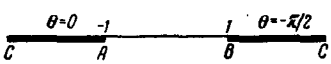

Fig. 2.

Then from (2.4) we obtain

$$V^2 - 2a\frac{d\theta}{d\varphi}qV - 1 = 0$$

Since $V_{a=0} = 1$, we have

$$V = aq\frac{d\theta}{d\varphi} + \sqrt{1 + a^2\left(q\frac{d\theta}{d\varphi}\right)^2} \tag{2.6}$$

We can try to solve the problem by the method of successive approximations, taking a to be a small parameter. In this paper only the first two approximations will be considered.

3. Solution of the problem in the absence of capillary forces ($a = 0$). To solve the problem to at least the second approximation, the problem must be solved first to the first approximation in the absence of capillary forces. The solution of the problem is well known. It may be found in any book on hydrodynamics which contains even a cursory account of the theory of jets. For the sake of brevity in the present paper we shall state this solution with derivation in a form suitable for subsequent treatment.

In order to obtain a general solution of the problem it is sufficient to map the regions of variation of the complex potential w and the Zhukovskii function ω into the upper half-plane of the parametric variable t (Fig. 2)

$$\omega = \ln \frac{dw}{v_0\, dz} = \ln V - i\theta$$

The region of variation of the complex potential w is represented by a stripe of width q. (Let $\psi = q$ on CBA, then we have $\psi = 0$ along the wall CA). From the conformal transformation of this strip onto the half-plane we readily obtain

$$w(t) = -\frac{q}{\pi} \ln (t+1) + iq \tag{3.1}$$

Formula (3.1) may be verified directly. Differentiating (3.1), we find

$$\frac{dw}{dt} = -\frac{q}{\pi(1+t)} \tag{3.2}$$

If the width of the opening is not varied with transition to subsequent approximations, the flow rate in the jet will necessarily be different in different approximations. It is convenient when proceeding to the next approximation to keep the flow q fixed and correspondingly to vary the width of the opening. Then the quantity q in (3.1) will be the same in all approximations.

The regions of variation of w and z will be different for different approximations. We define

$$(z)_{a=0} = z_1, \qquad (\omega)_{a=0} = \omega_1 = \ln (v_1 / v_0) - i\theta_1$$

On CA ($t < -1$) we have $\operatorname{Im} \omega_1 = -\theta = 0$; on AB ($-1 < t < 1$) the function ω_1 is a pure imaginary quantity, on BC ($t > 1$) $\operatorname{Im} \omega_1 = -\theta_1 = \pi/2$.

The function $\omega_1(t)$ has the form

$$\omega_1 = \ln \frac{\sqrt{1-t} + i\sqrt{1+t}}{\sqrt{2}}, \quad \text{or} \quad \frac{dw}{v_0\,dz_1} = \frac{\sqrt{1-t} + i\sqrt{1+t}}{\sqrt{2}} \tag{3.3}$$

Equations (3.3) may be verified directly. To do this it is sufficient to observe the variation of ω_1 and dw/v_0dz_1 along the real part of the t-axis.

4. Determination of ω for a given V. We find $\omega(t)$, assuming that in the interval $-1 < t < 1$ of the real axis the dependence of $V(t)$ is known. Let us introduce an auxiliary function:

$$\Omega = \omega - \omega_1 = \ln \frac{dw}{v_0 dz} - \ln \frac{\sqrt{1-t} + i\sqrt{1+t}}{\sqrt{2}} \tag{4.1}$$

The function Ω may be found using by the methods of thin-wing theory [5]. The boundary conditions on the real t-axis for ω, ω_1 and Ω are presented in tabular form:

t	ω	ω_1	Ω
$t>1$	$\mathrm{Im}\,\omega=\pi/2$	$\mathrm{Im}\,\omega_1=\pi/2$	$\mathrm{Im}\,\Omega=0$
$-1<t<1$	$\mathrm{Re}\,\omega=\ln V$	$\mathrm{Re}\,\omega_1=0$	$\mathrm{Re}\,\Omega=\ln V$
$t<-1$	$\mathrm{Im}\,\omega=0$	$\mathrm{Im}\,\omega_1=0$	$\mathrm{Im}\,\Omega=0$

The function $\Omega(1-t^2)^{-1/2}$ vanishes for $t \to \infty$ and on the real t-axis its real part is known everywhere. Therefore we can determine it by the Schwarz formula for the upper half-plane, which in accordance with the table of boundary values of $\Omega(t)$ gives

$$\Omega(t) = \frac{\sqrt{1-t^2}}{\pi i} \int_{-1}^{1} \frac{\ln V(\xi)\,d\xi}{\sqrt{1-\xi^2}\,(\xi - t)} \tag{4.2}$$

Then separating the imaginary part of (4.2), expressing $\ln V(\xi)$ in terms of $d\theta/d\phi$ with the help of (2.6) and eliminating then $d\phi$ (see (3.2)), it is easy to obtain for θ the integral-differential equation

$$\theta + \tan^{-1}\frac{\sqrt{1+t}}{\sqrt{1-t}} = \frac{\sqrt{1-t^2}}{\pi}\,\text{v. p.}\int_{-1}^{1} \frac{\ln\left[aq\,d\theta/d\varphi + \sqrt{1 + a^2q^2(d\theta/d\varphi)^2}\right]}{\sqrt{1-\xi^2}\,(\xi - t)}\,d\xi$$

$$\left(d\varphi = -\frac{q\,d\xi}{\pi(1+\xi)}\right)$$

The solution of the problem will now be presented in second approximation without, however, resorting to the use of this equation.

5. Solution of the problem in the second approximation. To find the Zhukovskii function in the second approximation with the help of (4.2) it is first necessary to find $V(t)$ for this case. When neglecting the quantities of order a^2 in (2.6), assuming $\theta = \theta_1$ and using (3.2), we find

$$V_2(t) = 1 + \frac{d\theta_1}{d\varphi} aq = 1 + \frac{\pi a}{2} \sqrt{\frac{1+t}{1-t}}$$

From this and from (4.2) we obtain in second approximation

$$\Omega = \frac{\sqrt{1-t^2}}{\pi i} \int_{-1}^{1} \frac{\ln [1 + a_1 \sqrt{1+\xi} / \sqrt{1-\xi}]}{\sqrt{1-\xi^2}(\xi - t)} d\xi \qquad \left(a_1 = \frac{\pi a}{2}\right) \tag{5.1}$$

In order to calculate the integral in Equation (5.1) we differentiate the latter with respect to the parameter a_1. We obtain

$$\frac{\partial \Omega}{\partial a_1} = \frac{\sqrt{1-t^2}}{\pi i} \int_{-1}^{1} \frac{d\xi}{[1 + a_1 \sqrt{1+\xi} / \sqrt{1-\xi}](\xi - t)(1-\xi)}$$

Making the substitution $(1 + \xi)^{1/2}(1 - \xi)^{-1/2} = \eta$, we can reduce the integral to the integral of a rational fraction. The calculation of this integral yields

$$\frac{\partial \Omega}{\partial a_1} = \frac{\sqrt{1+t}}{\sqrt{1-t} + a_1 \sqrt{1+t}} + \frac{2i}{\pi} \sqrt{1-t^2} \frac{\ln [a_1 \sqrt{1+t} / \sqrt{1-t}]}{1 - t - a_1^2(1+t)}$$

Hence

$$\Omega = \ln\left(1 + a_1 \frac{\sqrt{1+t}}{\sqrt{1-t}}\right) + \frac{2i}{\pi} \sqrt{1-t^2}\, \vartheta(a_1, t)$$

$$\vartheta(a_1, t) = \int_0^{a_1} \frac{\ln [a_1 \sqrt{1+t} / \sqrt{1-t}]}{1 - t - a_1^2(1+t)} da_1 \tag{5.2}$$

6. Calculation of the coefficient of contraction of the jet. From Figs. 1 and 2 we have

$$l - \delta = \int_{t=-1}^{t=1} dy \tag{6.1}$$

In order to find the contraction coefficient of the jet $k = \delta/l$ we calculate the integral on the right-hand side of (6.1). From (4.1) it follows that

$$\frac{dw}{v_0\, dz} = \frac{\sqrt{1-t} + i\sqrt{1+t}}{\sqrt{2}} e^{\Omega}$$

From this, by the use of (3.2), we find

$$dz = -\frac{q e^{-\Omega} \sqrt{2}}{\pi v_0 (\sqrt{1-t} + i\sqrt{1+t})(1+t)} dt \tag{6.2}$$

Or upon using (5.2) and making simple transformations

$$dz = -\frac{\delta(\sqrt{1-t} - i\sqrt{1+t})}{\pi\sqrt{2}(1+a_1\sqrt{1+t}/\sqrt{1-t})(1+t)}\exp\left(-\frac{2i}{\pi}\sqrt{1-t^2}\vartheta(a_1, t)\right)dt \quad (6.3)$$

Separating in (6.3) the real and imaginary parts and neglecting as small the quantities in terms of higher orders of a_1, we have

$$dy = \frac{\delta}{\pi\sqrt{2}}\left[\frac{1}{\sqrt{1+t}(1+a_1\sqrt{1+t}/\sqrt{1-t})} + \right.$$
$$\left. + \frac{\sqrt{1-t}}{(1+t)(1+a_1\sqrt{1+t}/\sqrt{1-t})}\frac{2}{\pi}\sqrt{1-t^2}\,\vartheta(a_1, t)\right]dt \quad (6.4)$$

Substituting dy from (6.3) into (6.1), we obtain

$$l - \delta = \frac{\delta}{\pi\sqrt{2}}\{J_1 + J_2\}, \text{ where } J_1 = \int_{-1}^{1}\frac{dt}{\sqrt{1+t}(1+a_1\sqrt{1+t}/\sqrt{1-t})} \quad (6.5)$$

$$J_2 = \frac{2}{\pi}\int_{-1}^{1}\frac{(1-t)\,\vartheta(a_1,t)}{\sqrt{1+t}\,(1+a_1\sqrt{1+t}/\sqrt{1-t})}\,dt \quad (6.6)$$

The integral J_1 is not difficult to calculate exactly, when it is reduced to an integral of a rational fraction. After discarding in the expression obtained the quantities of higher order in a_1, we obtain

$$J_1 = 2\sqrt{2}(1 - a_1) + O(a_1^2 \ln a_1) \quad (6.7)$$

Excluding an infinitely small region about the point $t = 1$, the following approximation is used for the calculation of the integral J_2:

$$\frac{\vartheta(a_1, t)}{1 + a_1\sqrt{1+t}/\sqrt{1-t}} \approx \frac{1}{1-t}\int_0^{a_1}\left[\ln a_1 + \ln\sqrt{\frac{1+t}{1-t}}\right]da_1$$

Hence

$$J_2 \approx \frac{4a_1\sqrt{2}}{\pi}[\ln a_1 - 1 - \ln 2]$$

Thus, from Equation (6.4) we obtain that

$$l - \delta \approx \frac{2\delta}{\pi}\left\{1 + \frac{2a_1 \ln a_1}{\pi} - a_1\left[1 + \frac{2}{\pi} + \frac{2}{\pi}\ln 2\right]\right\}$$

and the coefficient $k(a)$ in terms of the new parameter $a = 2a_1/\pi$ is

$$k(a) = \frac{\pi}{\pi+2}\left[1 + 2a\frac{\ln(2/\pi a)}{\pi+2} + a\left(1 + \frac{2\ln 2}{\pi+2}\right)\right] \quad (6.8)$$

The factor $\pi(\pi + 2) = k(0)$ is equal to the coefficient of contraction for the case of zero forces of capillarity. The parameter a may be represented in the form

$$a = \frac{\alpha}{\rho v_0^2 \delta} = \frac{\alpha}{\rho v_0^2 l k(a)} \approx \frac{\alpha}{\rho l v_0^2 k(0)} = \frac{\alpha(2+\pi)}{\rho l v_0^2 \pi}$$

Hence Equation (6.8) may be reduced to the form

$$k(a) = k(0)\left\{1 + \frac{\alpha}{\pi \rho l v_0^2}\left[2\ln\frac{2\rho l v_0^2}{\alpha(2+\pi)} + \pi + 2 + 2\ln 2\right]\right\} \qquad (6.9)$$

or

$$k(a) \approx 0.611\left\{1 + \frac{0.318\alpha}{\rho l v_0^2}\left[2\ln\frac{0.39\rho l v_0^2}{\alpha} + 6.52\right]\right\} \qquad (6.10)$$

7. Capillary waves on the surface of a flow of finite depth. The solution derived above is not unique, because there may exist capillary waves on the free surface. It will be shown below how the flow with capillary waves of small amplitude may be superimposed on a given flow.

First of all we must investigate the behavior of the complex velocity at infinity, i.e. in the vicinity of point A. To this end let us consider the problem of capillary waves of small amplitude, which move along the surface of a fluid flow of finite depth (Fig. 3). The theory of capillary waves of small amplitude is a well-investigated part of hydrodynamics [6]; nevertheless we shall consider the solution of this problem briefly in order to reduce it to a form suitable for our purposes.

Consider sinusoidal waves of small amplitude. Let δ be an average depth of a flow and v_0 the velocity at some point of inflection D of the curved surface of the sinusoidal wave. The rate of flow q equals $v_0\delta$ to within terms of higher order. On a free surface Equations (2.1) and (2.2) must be satisfied. Since at the point $D R = \infty$, the atmospheric pressure p_1 at this point is equal to the pressure of the fluid, and Equation (2.4) may be utilized as a boundary condition on the free surface.

It is evident that at the bottom, which we shall consider to coincide with the x-axis, the vertical y-component of the velocity vanishes.

Let w be a complex potential. Let us look for a solution in the form

$$\frac{dw}{dZ} = v_0\left[1 - \varkappa A \sin\frac{\varkappa(w+\varphi_0)}{v_0}\right] \qquad (7.1)$$

where A, κ and ϕ_0 are real constants and $Z = X + iY$. Amplitude of the wave A is considered to be a small quantity.

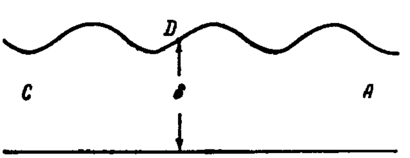

Fig. 3.

The values of w at the bottom are real and on the free surface $\operatorname{Im} w = \psi = v_0\delta$. It is seen from (7.1) that for real values of w the complex velocity $dw/dZ = v^\circ e^{-i\theta^\circ} = v_X - iv_Y$, is also real, i.e. the vertical velocity component is zero. It remains, therefore, to satisfy the boundary condition on the free surface $\psi = v_0\delta$.

Let us write the boundary condition (2.4) in a linearized form. When replacing θ by θ° and $v \approx v_0$ by v°, we obtain

$$\alpha \frac{d\theta^\circ}{d\varphi} = \rho (v^\circ - v_0) \tag{7.2}$$

Since on the free surface $w = \phi + iv_0\delta$

$$v^\circ e^{-i\theta^\circ} = v_0\left\{1 - \varkappa A \sin\left[\frac{\varkappa}{v_0}(\varphi + \varphi_0) + i\varkappa\delta\right]\right\}$$
$$= v_0\left\{1 - \varkappa A \sin\frac{\varkappa}{v_0}(\varphi + \varphi_0)\cosh\delta\varkappa - i\varkappa A\cos\frac{\varkappa}{v_0}(\varphi + \varphi_0)\sinh\varkappa\delta\right\}$$

From this, neglecting quantities containing higher orders of the small amplitude A, we find

$$v^\circ = v_0\left[1 - \varkappa A\sin\frac{\varkappa}{v_0}(\varphi + \varphi_0)\cosh\varkappa\delta\right], \quad \theta^\circ = \varkappa A\cos\frac{\varkappa}{v_0}(\varphi + \varphi_0)\sinh\varkappa\delta$$

Substituting these expressions for v° and θ° into (7.2), we obtain after obvious cancellations

$$\varkappa \tanh\varkappa\delta = \frac{\rho v_0^2}{\alpha} \tag{7.3}$$

Equation (7.3) determines the frequency κ, at which small sinusoidal capillary waves are possible. For relatively small a and sufficient depths $\tanh\kappa\delta \approx 1$ and $\kappa \approx \rho v_0^2/a \gg 1$.

The region of variation of the complex potential w wholly coincides with the region of variation of the complex potential w considered in the foregoing sections. Let us map the region w upon the upper half-place of the parametric variable t.

In this case Equation (3.1) may be used. We have

$$\frac{dw}{dZ} = v_0\left\{1 - \varkappa A\sin\left[\frac{\varkappa}{v_0}\left(-\frac{v_0\delta}{\pi}\ln(1+t) + iv_0\delta + \varphi_0\right)\right]\right\} \tag{7.4}$$

Let us introduce the function

$$\omega^\circ = \ln\frac{dw}{v_0\,dZ} = \ln V^\circ - i\theta^\circ \tag{7.5}$$

If we now add ω° to the previous Zhukovskii function $\omega = \omega_1 + \Omega$ and take the function $\omega = \omega_1 + \Omega + \omega^\circ$ as a new Zhukovskii function, then this function together with the complex potential w from (3.1) will determine a certain jet flow with capillary waves on its free surface. However, in this case the jet would not emanate from an opening in a plane, but from an opening in some curved wall. Let us therefore change the Zhukovskii function once more by adding to $\omega_1 + \Omega + w^\circ$ an additional term ω_+, correcting thereby the boundary conditions without changing the capillary waves at infinity. Therefore, let us consider the final Zhukovskii function and try to determine ω_+.

$$\omega = \omega_1 + \Omega + \omega^\circ + \omega_+ = \ln V - i\theta \tag{7.6}$$

8. Determination of the additional function $\omega_+ = V_+ - i\theta_+$. The function ω_+ is holomorphic in the upper half-plane of the parametric variable t. Let us find the boundary conditions for it on the real t-axis.

At the bottom AC the real values of w are valid, where $t < -1$ and we have

$$\operatorname{Im}\left[-\frac{v_0\delta}{\pi}\ln(1+t) + iv_0\delta\right] = \operatorname{Im}\left[-\frac{v_0\delta}{\pi}\ln(-1-t)\right] = 0$$

Hence, in accordance with Equations (7.4) and (7.5) we have $\operatorname{Im}\omega^\circ = \theta^\circ = 0$. Since along the bottom necessarily $\operatorname{Im}\omega = 0$, it follows from this and from (7.6), in accordance with the table of the boundary values of Section 4, that

$$\theta_+ = 0 \qquad \text{for } t < -1 \tag{8.1}$$

On the wall $BC(t > 1)$ we have $\theta = -\operatorname{Im}\omega = -\pi/2$. From (7.4) and (7.5), neglecting higher orders of the small amplitude, we find

$$\theta^\circ = \varkappa A \cos\left[-\frac{\varkappa\delta}{\pi}\ln(1+t) + \frac{\varkappa\varphi_0}{v_0}\right]\sinh \varkappa\delta$$

This equation together with the values of the imaginary parts of ω_1 and Ω (see Table in Section 4) gives for θ_+ the boundary condition

$$-\operatorname{Im}\omega_+ = \theta_+ = -\varkappa A\cos\left[-\frac{\varkappa\delta}{\pi}\ln(1+t) + \frac{\varkappa\varphi_0}{v_0}\right]\sinh\varkappa\delta \quad \text{for } t > 1 \tag{8.2}$$

Next consider the boundary condition (2.4) on the free surface. In a linearized form it may be written as follows (compare with (7.2)):

$$\alpha\frac{d\theta}{d\varphi} = \rho v_0 (V - 1) \tag{8.3}$$

It is evident that

$$\frac{d\theta}{d\varphi} = \frac{d\theta_1}{d\varphi} + \frac{d\theta_2}{d\varphi} + \frac{d\theta^\circ}{d\varphi} + \frac{d\theta_+}{d\varphi}$$

where $d\theta_2/d\phi$ in accordance with the chosen method of solution is a small quantity in comparison with $d\theta_1/d\phi$.

Condition (7.3) shows that κ is large, i.e. that the frequency of the capillary waves is large. Therefore, in differentiating with respect to ϕ there appears a new large factor κ.

Let us assume that θ_+ varies uniformly together with ϕ and approaches zero when $\phi \to \infty$. Therefore, now we may assume that $d\theta_+/d\phi$ is small in comparison with $d\theta^\circ/d\phi$. Hence

$$\frac{d\theta}{d\varphi} \approx \frac{d\theta_1}{d\varphi} + \frac{d\theta^\circ}{d\varphi} \tag{8.4}$$

On the other hand

$$\ln V = \ln V_1 + \ln V_2 + \ln V^\circ + \ln V_+, \qquad \text{or} \qquad V = V_1 V_2 V^\circ V_+$$

Consequently, in accordance with the solutions obtained above (see Sections 5 and 4) and Equations (7.2), we find

$$V = 1\left(1 + aq\,\frac{d\theta_1}{d\varphi}\right)\left(1 + \frac{\alpha}{\rho v_0}\,\frac{d\theta^\circ}{d\varphi}\right)V_+ \tag{8.5}$$

since according to (2.5) $aq = \alpha/\rho v_0$ and for small $V_+ - 1$ we have $V_+ \approx 1 + \ln V_+$, then from (8.3), (8.4) and (8.5) we find up to higher-order terms

$$\alpha\left(\frac{d\theta_1}{d\varphi} + \frac{d\theta^\circ}{d\varphi}\right) = \rho v_0\left(\frac{\alpha}{\rho v_0}\,\frac{d\theta_1}{d\varphi} + \frac{\alpha}{\rho v_0}\,\frac{d\theta^\circ}{d\varphi} + \ln V_+\right)$$

From this we obtain the boundary conditions on the free surface

$$\ln V_+ = 0 \quad \text{for } -1 < t < 1 \tag{8.6}$$

Thus, the problem of determination of ω_+ reduces to the known problem of finding the function of a complex variable, holomorphic in the upper half-plane subject to the condition that Im ω_+ of the function (see (8.1)) and (8.2) is given along a portion of the boundary, and that Re ω_+ (see (8.6)) is given on another portion of the boundary.

In order to solve this problem we introduce the auxiliary function $\omega_+/\sqrt{(t^2-1)}$. In the case of this function its imaginary part will be given on the real t-axis. The Schwarz equation for the upper-half-plane gives

$$\frac{\omega_+}{\sqrt{t^2-1}} = \operatorname{sh}\varkappa\delta\,\frac{\varkappa A}{\pi}\int_1^\infty \frac{\cos\left[(-\varkappa\delta/\pi)\ln(1+\xi) + \varkappa\varphi_0/v_0\right]d\xi}{\sqrt{\xi^2-1}\,(\xi - t)} \tag{8.7}$$

In the expression for ω and ω_+ there are two unknown constants A and ϕ_0.

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